The Lie algebraic structure of extended Sutherland models

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Abstract

We disclose the Lie algebraic structure of two extended Sutherland models. Their Hamiltonians are BC_N, and A_N Sutherland Hamiltonians with some additional terms. We show that both Hamiltonians can be written in the quadratic forms of generators of the Lie algebra $gl(N+1)$.

Keywords : Quasi-exactly-solvability; Sutherland models; Lie algebra.

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1 Introduction

The quasi-exactly-solvable models have been developed considerably over last decade. Ten years ago the crucial definition of quasi-exact-solvability was made.[1] The author defined the quasi-exact-solvability as follows. Let $\mathcal{P}_n$ be the space of polynomials of degree $\leq n$.

Definition 1 *Let us name a linear differential operator of the k-th order,* T_k quasi-exactly-solvable, *if it preserves the space* $\mathcal{P}_n$. *Correspondingly, the operator* E_k, *which preserves the infinite flag* $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_n \subset \cdots$ *of spaces of all polynomials, is named* exactly-solvable.

In particular, if operators can be represented by the elements of the enveloping algebra of a certain Lie algebras realized in terms of differential operators preserving the space $\mathcal{P}_n$, they are quasi-exactly-solvable. Following this concept, various quasi-exactly-solvable models have been proposed. In reference,[2] the authors obtain the sl_2-based deformation of the Calogero models. A_N, BC_N, B_N, C_N, and D_N Calogero and Sutherland Hamiltonian, as well as thier supersymmetric generalizations are shown to have expression in the quadratic polynomials in the generators of the Lie algebra or the Lie superalgebra for the supersymmetric case.[3] Moreover the $sl(N+1)$ deformation of the Calogero models, whose Hamiltonians have the quadratic and sextic self-interaction terms, are shown to be quasi-exact-solvable.[4] Recently the quasi-exact-solvable models are classified

by using the new family of A_N-type Dunkl operators which includes the usual Dunkl operator.[5]

The purpose of this paper is to obtain the explicit Lie algebraic form of the extended Sutherland Hamiltonians, following the method explained in the reference.[3] We consider here two kinds of Hamiltonians. One of them is an extended Hamiltonian of the BC_N-Sutherland model, which has an additional term:

$$\sum_{i=1}^{N} \left\{ g_1 \cos^2 2x_i + 2g_2 \cos 2x_i \right\}. \tag{1}$$

The other is an extended Hamiltonian of the A_N-Sutherland model, which has an additional term:

$$\sum_{i=1}^{N} \{ g_1 \cos 4x_i + g_2 \cos 2x_i + g_3 \sin 4x_i + g_4 \sin 2x_i \}. \tag{2}$$

These Hamiltonians are indicated in the reference.[5]

This paper is arranged as follows. In section 2, we consider an extended BC_N-Sutherland Hamiltonian. First we rotate it with the ground state eigenfunction as a gauge fuctor. Then we rewrite the gauge rotated Hamiltonian in terms of the elementary symmetric polynomials. Finally we arrive at the explicit form of the Hamiltonian in quadratic form of generators which generate the Lie algebra $gl(N+1)$. In section 3, we show an extended A_N-Sutherland Hamiltonian has Lie algebraic form as the same technique as in section 2.

2 Extended BC_N-Sutherland model

In this section we begin with the case of the extended BC_N-Sutherland Hamiltonian.

2.1 *Hamiltonian*

The Hamiltonian for the Sutherland model of type BC_N is[3]

$$\begin{aligned} H &= -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \sum_{i \neq j} \left\{ \frac{1}{\sin^2(x_i - x_j)} + \frac{1}{\sin^2(x_i + x_j)} \right\} \\ &\quad + 4g_3 \sum_{i=1}^{N} \frac{1}{\sin^2 2x_i} + g_4 \sum_{i=1}^{N} \frac{1}{\sin^2 x_i} \end{aligned} \tag{3}$$

where g, g_3 and g_4 are coupling constants. From the Hamiltonian (3) we can deduce the Hamiltonian of the Sutherland models of type B_N, C_N, D_N as follows:

$$\begin{aligned} g_3 = 0 &\quad : \quad B_N, \\ g_4 = 0 &\quad : \quad C_N, \\ g_3 = g_4 = 0 &\quad : \quad D_N. \end{aligned}$$

We will consider a Hamiltonian with the external potential[5]

$$\sum_{i=1}^{N} \left\{ g_1 \cos^2 2x_i + 2g_2 \cos 2x_i \right\}, \tag{4}$$

that is,

$$
\begin{aligned}
H &= -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g\sum_{i\neq j}\left\{\frac{1}{\sin^2(x_i-x_j)}+\frac{1}{\sin^2(x_i+x_j)}\right\} \\
&\quad + \sum_{i=1}^{N}\left\{g_1\cos^2 2x_i + 2g_2\cos 2x_i + 4g_3\frac{1}{\sin^2 2x_i} + g_4\frac{1}{\sin^2 x_i}\right\}
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
g &= \nu(\nu-1), \\
g_1 &= -\nu_1^2, \\
g_2 &= \nu_1\left\{1+2n+2(N-1)\nu+\nu_3+\nu_4\right\} \quad (n\in\mathbf{Z}), \\
g_3 &= -\nu_3(\nu_3-1), \\
g_4 &= \nu_3(\nu_3-1)-\nu_4(\nu_4-1).
\end{aligned}
$$

Let us call the Hamiltonian (5) the extended BC_N-Sutherland Hamiltonian.

2.2 *Gauge rotation*

First we make a gauge rotation of the Hamiltonian (5) with the ground state eigenfunction

$$
\begin{aligned}
\mu &= \left\{\prod_{i<j}\sin(x_i-x_j)\sin(x_i+x_j)\right\}^{\nu} \\
&\quad \times\left\{\prod_{i=1}^{N}\cos x_i\right\}^{\nu_3}\left\{\prod_{i=1}^{N}\sin x_i\right\}^{\nu_4}\prod_{i=1}^{N}\exp\left(-\frac{\nu_1}{2}\cos 2x_i\right)
\end{aligned} \tag{6}
$$

as gauge factor, $\widetilde{H}=\mu^{-1}H\mu$. Using the relation

$$
\begin{aligned}
&\sum_{i=1}^{N}\left[\sum_{j(\neq i)}\left(\frac{\cos(x_i-x_j)}{\sin(x_i-x_j)}+\frac{\cos(x_i+x_j)}{\sin(x_i+x_j)}\right)\right]^2 \\
&= 2\sum_{i<j}\left(\frac{\cos^2(x_i-x_j)}{\sin^2(x_i-x_j)}+\frac{\cos^2(x_i+x_j)}{\sin^2(x_i+x_j)}\right) + \mathit{constant},
\end{aligned} \tag{7}
$$

it is not difficult to obtain that

$$
\begin{aligned}
\widetilde{H} &= -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} - 2\sum_{i=1}^{N}\left\{\nu\sum_{j(\neq i)}\left(\frac{\cos(x_i-x_j)}{\sin(x_i-x_j)}+\frac{\cos(x_i+x_j)}{\sin(x_i+x_j)}\right)\right. \\
&\qquad \left. +(\nu_3+\nu_4)\frac{\cos 2x_i}{\sin 2x_i} - (\nu_3-\nu_4)\frac{1}{\sin 2x_i} + \nu_1\sin 2x_i\right\}\frac{\partial}{\partial x_i} \\
&\qquad +4n\nu_1\sum_{i=1}^{N}\cos 2x_i,
\end{aligned} \tag{8}
$$

hereafter we will omit the constant terms.

2.3 *Lie algebraic form*

In order to disclose the Lie algebraic structure of the gauge rotated Hamiltonian (8), it is expedient to rewrite the Hamiltonian $\widetilde{H}$ in terms of the elementary symmetric polynomials. Let e_i be the i-th elementary symmetric polynomials of $\cos 2x$,

$$\begin{aligned}
e_1 &= \sum_{i=1}^{N} \cos 2x_i, \\
e_2 &= \sum_{i<j} \cos 2x_i \cos 2x_j, \\
e_3 &= \sum_{i<j<k} \cos 2x_i \cos 2x_j \cos 2x_k, \\
&\vdots \\
e_N &= \cos 2x_1 \cos 2x_2 \cdots \cos 2x_N.
\end{aligned}$$

Lemma 1 *In these variables $\{e_1, e_2, \ldots, e_N\}$, the gauge rotated Hamiltonian $\widetilde{H}$ becomes*

$$\begin{aligned}
\widetilde{H} = & -4 \sum_{k,l=1}^{N} \Bigg\{ N e_{k-1} e_{l-1} - \sum_{i \geq 0} \Big[(k-i) e_{k-i} e_{l+i} + (l-1+i) e_{k-1-i} e_{l-1+i} \\
& \quad - (k-2-i) e_{k-2-i} e_{l+i} - (l+1+i) e_{k-1-i} e_{l+1+i} \Big] \Bigg\} \frac{\partial}{\partial e_k} \frac{\partial}{\partial e_l} \\
& + 4 \sum_{k=1}^{N} \Bigg\{ \nu (N-k+2)(N-k+1) e_{k-2} \\
& \quad + \Big[\nu k (2N-k-1) + 2\nu_3 k + k \Big] e_k \\
& \quad + \nu_1 \Big[-e_1 e_k + (N-k+1) e_{k-1} + (k+1) e_{k+1} \Big] \Bigg\} \frac{\partial}{\partial e_k} \\
& + 4 n \nu_1 e_1.
\end{aligned} \tag{9}$$

Proof. After the change of variables $x_i \to e_i$, the Hamiltonian (8) is

$$\begin{aligned}
\widetilde{H} = & - \sum_{k,l=1}^{N} \sum_{i=1}^{N} \frac{\partial e_k}{\partial x_i} \frac{\partial e_l}{\partial x_i} \frac{\partial}{\partial e_k} \frac{\partial}{\partial e_l} - \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{\partial^2 e_k}{\partial x_i^2} \frac{\partial}{\partial e_k} \\
& - \nu \sum_{k=1}^{N} \sum_{i \neq j} \Bigg\{ \frac{\cos(x_i - x_j)}{\sin(x_i - x_j)} \left(\frac{\partial e_k}{\partial x_i} - \frac{\partial e_k}{\partial x_j} \right) \\
& \quad + \frac{\cos(x_i + x_j)}{\sin(x_i + x_j)} \left(\frac{\partial e_k}{\partial x_i} + \frac{\partial e_k}{\partial x_j} \right) \Bigg\} \frac{\partial}{\partial e_k} \\
& - 2(\nu_3 + \nu_4) \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{\cos 2x_i}{\sin 2x_i} \frac{\partial e_k}{\partial x_i} \frac{\partial}{\partial e_k}
\end{aligned} \tag{10}$$

$$+2(\nu_3-\nu_4)\sum_{k=1}^{N}\sum_{i=1}^{N}\frac{1}{\sin 2x_i}\frac{\partial e_k}{\partial x_i}\frac{\partial}{\partial e_k}$$
$$-2\nu_1\sum_{k=1}^{N}\sum_{i=1}^{N}\sin 2x_i\frac{\partial e_k}{\partial x_i}\frac{\partial}{\partial e_k}+4n\nu_1\sum_{i=1}^{N}\cos 2x_i.$$

As an example, let us demonstrate how to express

$$X_k=\sum_{i=1}^{N}\sin 2x_i\frac{\partial e_k}{\partial x_i} \tag{11}$$

in terms of the elementary symmetric polynomials. Let us consider the generating function of X_k:

$$X(t)=\sum_{k=0}^{N}X_k t^k.$$

Note that the generating function of elementary symmetric polynomials e_k is

$$E(t)=\sum_{k=0}^{N}e_k t^k=\exp\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}p_n(\cos 2x)t^n\right], \tag{12}$$

where p_n are the power sums:

$$p_n(\cos 2x)=\sum_{i=1}^{N}\cos^n 2x_i.$$

Calculating the generating function $X(t)$, we have

$$\begin{aligned} X(t) &= \sum_{k=0}^{N}\left[\sum_{i=1}^{N}\sin 2x_i\frac{\partial e_k}{\partial x_i}\right]t^k \\ &= \sum_{i=1}^{N}\sin 2x_i\sum_{k=0}^{N}\frac{\partial e_k}{\partial x_i}t^k \\ &= \sum_{i=1}^{N}\sin 2x_i\frac{\partial E(t)}{\partial x_i} \\ &= 2\left\{\sum_{n=1}^{\infty}(-1)^n p_{n-1}(\cos 2x)t^n-\sum_{n=1}^{\infty}(-1)^n p_{n+1}(\cos 2x)t^n\right\}E(t). \end{aligned} \tag{13}$$

It is not difficult to show that

$$X(t)=2\left(p_1(\cos 2x)-Nt+t^2\frac{\partial}{\partial t}-\frac{\partial}{\partial t}\right)E(t). \tag{14}$$

Substituting $E(t)=\sum_{k=0}^{N}e_k t^k$ in (14), we find that

$$X_k=2\left\{e_1e_k-(N-k+1)e_{k-1}-(k+1)e_{k+1}\right\}. \tag{15}$$

One can show the other parts similarly. □

Next we will realize the Lie algebra $gl(N+1)$ in terms of elementary symmetric polynomials. One of the simplest realization of $gl(N+1)$ is the differential realization. The generators can be represented in the following form:

$$J_i^- = \frac{\partial}{\partial e_i} \qquad (i = 1, 2, \dots, N), \tag{16}$$

$$J_{ij}^0 = e_i \frac{\partial}{\partial e_j} \qquad (i, j = 1, 2, \dots, N), \tag{17}$$

$$J^0 = n - \sum_{i=1}^{N} e_i \frac{\partial}{\partial e_i}, \tag{18}$$

$$J_i^+ = e_i J^0 \qquad (i = 1, 2, \dots, N), \tag{19}$$

where the parameter n is an integer.

If n is a non-negative integer, the generators act on the representation space of polynomials in N variables of the following type

$$\mathcal{P}_n = \mathrm{span}\left\{ e_1^{n_1} e_2^{n_2} \cdots e_N^{n_N} \;\middle|\; 0 \le \sum_{i=1}^{N} n_i \le n \right\}. \tag{20}$$

With simple algebraic transformations from (9), we obtain the following result.

Proposition 1 *We obtain the Lie algebraic form of the gauge rotated extended BC_N-Sutherland Hamiltonians (9) as follows:*

$$\begin{aligned}
&\widetilde{H} = \\
&-4 \sum_{k,l=1}^{N} \Bigg\{ N J_{k-1,l}^0 J_{l-1,k}^0 - \sum_{i \ge 0} \Bigg[(k-i)\big(J_{k-i,l}^0 J_{l+i,k}^0 - \delta_{i0} J_{k,k}^0\big) \\
&\quad + (l-1+i)\big(J_{k-1-i,l}^0 J_{l-1+i,k}^0 - \delta_{i1} J_{k-2,k}^0\big) \\
&\quad - (k-2-i)\big(J_{k-2-i,l}^0 J_{l+i,k}^0 - \delta_{i0} J_{k-2,k}^0\big) - (l+1+i) J_{k-1-i,l}^0 J_{l+1+i,k}^0 \Bigg] \Bigg\} \\
&+4 \sum_{k=1}^{N} \Bigg\{ \nu(N-k+2)(N-k+1) J_{k-2,k}^0 + \nu_1(N-k+1) J_{k-1,k}^0 \\
&\quad + \Big[\nu k(2N-k-1) + 2\nu_3 k + k \Big] J_{k,k}^0 + \nu_1(k+1) J_{k+1,k}^0 \Bigg\} \\
&+4\nu_1 J_1^+.
\end{aligned} \tag{21}$$

Here we define

$$J_{0i}^0 = J_i^- \qquad (i = 1, 2, \dots, N).$$

3 Extended A_N-Sutherland Hamiltonian

In this section we devote to a consideration of the extended A_N-Sutherland Hamiltonian with using the same approach as in the previous section.

3.1 *Hamiltonian*

A second Hamiltonian we deal with is an extension of A_N-Sutherland model:

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \sum_{i \neq j} \frac{1}{\sin^2(x_i - x_j)}. \tag{22}$$

Considering the external potential[5]

$$2 \sum_{i=1}^{N} \{ g_1 \cos 4x_i + 2g_2 \cos 2x_i + 2g_3 \sin 4x_i + 2g_4 \sin 2x_i \}, \tag{23}$$

we define a Hamiltonian

$$\begin{aligned} H &= -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \sum_{i \neq j} \frac{1}{\sin^2(x_i - x_j)} \\ &\quad + 2 \sum_{i=1}^{N} \{ g_1 \cos 4x_i + 2g_2 \cos 2x_i + 2g_3 \sin 4x_i + 2g_4 \sin 2x_i \} \end{aligned} \tag{24}$$

where coupling constants g, g_1, g_2, g_3, g_4 satisfy

$$\begin{aligned} g &= \nu(\nu - 1), \\ g_1 &= (\beta - \gamma)(\beta + \gamma), \\ g_2 &= (\gamma + \alpha\beta + n\gamma + \nu\gamma(N-1)), \\ g_3 &= \beta\gamma, \\ g_4 &= (\alpha\gamma - \beta - n\beta - \nu\beta(N-1)). \end{aligned}$$

We call the Hamiltonian (24) the extended A_N-Sutherland Hamiltonian.

3.2 *Gauge rotation*

Similarly to what was done in section 2 for the extended BC_N-Sutherland Hamiltonian, we first make a gauge rotation of (24) with the ground state eigenfunction

$$\mu(x) = \prod_{i<j} \sin^{\nu}(x_i - x_j) \prod_{i=1}^{N} \cos^n x_i \exp\left[\alpha x_i + \beta \sin 2x_i - \gamma \cos 2x_i\right] \tag{25}$$

as a gauge fuctor $\widetilde{H} = \mu^{-1} H \mu$. It is not difficult to obtain

$$\begin{aligned} \widetilde{H} &= -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i=1}^{N} \Bigg\{ \nu \sum_{i \neq j} \frac{\cos(x_i - x_j)}{\sin(x_i - x_j)} - n \frac{\sin x_i}{\cos x_i} \\ &\qquad + (\alpha + 2\beta \cos 2x_i + 2\gamma \sin 2x_i) \Bigg\} \frac{\partial}{\partial x_i} \\ &\quad - n(n-1) \sum_{i=1}^{N} \tan^2 x_i + 2\nu n \sum_{i<j} \tan x_i \tan x_j \\ &\quad + 2n(\alpha - 2\beta) \sum_{i=1}^{N} \tan x_i. \end{aligned} \tag{26}$$

3.3 *Lie algebraic form*

Next we rewrite the gauge rotated Hamiltonian (26) in terms of the elementary symmetric polynomials of $\tan x$.

$$\begin{aligned} e_1 &= \sum_{i=1}^{N} \tan x_i, \\ e_2 &= \sum_{i<j} \tan x_i \tan x_j, \\ e_3 &= \sum_{i<j<k} \tan x_i \tan x_j \tan x_k, \\ &\vdots \\ e_N &= \tan x_1 \tan x_2 \cdots \tan x_N. \end{aligned}$$

Numerous but straightforward calculations similar to what we made in the proof of lemma 1 leads to the following representation.

$$\begin{aligned} \widetilde{H} \;=\; & -\sum_{k,l=1}^{N} \Bigg\{ N e_{k-1} e_{l-1} - e_1 e_{k+1} e_l - e_1 e_k e_{l+1} + (e_1^2 - 2e_2) e_k e_l \\ & + \sum_{i\geq 0} \Bigg[(k-2-i) e_{k-2-i} e_{l+i} - (l-1+i) e_{k-1-i} e_{l-1+i} \\ & \quad +2(k-i) e_{k-i} e_{l+i} - 2(l+1+i) e_{k-1-i} e_{l+1+i} \\ & \quad +(k+2-i) e_{k+2-i} e_{l+i} - (l+3+i) e_{k-1-i} e_{l+3+i} \Bigg] \Bigg\} \frac{\partial}{\partial e_k} \frac{\partial}{\partial e_l} \\ & +\nu \sum_{k=1}^{N} \Bigg\{ (N-k+2)(N-k+1) e_{k-2} \\ & \quad -2 \Big[k(N-k) + e_2 \Big] e_k + (k+1) k e_{k+2} \Bigg\} \frac{\partial}{\partial e_k} \\ & -\sum_{k=1}^{N} \Bigg\{ 2 \Big[(k+2) e_{k+2} - e_1 e_{k+1} + (k + e_1^2 - 2e_2) e_k \Big] \Bigg\} \frac{\partial}{\partial e_k} \\ & +2n \sum_{k=1}^{N} \Bigg\{ (k+2) e_{k+2} - e_1 e_{k+1} + (k + e_1^2 - 2e_2) e_k \Bigg\} \frac{\partial}{\partial e_k} \\ & -2\alpha \sum_{k=1}^{N} \Bigg\{ \Big[N - (k-1) \Big] e_{k-1} + e_1 e_k - (k+1) e_{k+1} \Bigg\} \frac{\partial}{\partial e_k} \\ & -4\beta \sum_{k=1}^{N} \Bigg\{ \Big[N - (k-1) \Big] e_{k-1} - e_1 e_k + (k+1) e_{k+1} \Bigg\} \frac{\partial}{\partial e_k} \\ & -4\gamma \sum_{k=1}^{N} 2k e_k \frac{\partial}{\partial e_k} \\ & -n(n-1)(e_1^2 - 2e_2) + 2\nu n e_2 + 2n(\alpha - 2\beta) e_1. \end{aligned} \tag{27}$$

Now we use the same realization (16)-(19) of Lie algebra $gl(N+1)$. With simple algebraic transformations from (27), we derive the following result.

Proposition 2 *We obtain the Lie algebraic form of the gauge rotated extended A_N-Sutherland Hamiltonian (27) as follows:*

$$
\begin{aligned}
&\widetilde{H} = \\
&-\sum_{k,l=1}^{N}\Bigg\{ N J^0_{k-1,l}J^0_{l-1,k} + \sum_{i\geq 0}\Big[(k-2-i)(J^0_{k-2-i,l}J^0_{l+i,k} - \delta_{i0}J^0_{k-2,k}) \\
&\quad -(l-1+i)(J^0_{k-1-i,l}J^0_{l-1+i,k} - \delta_{i1}J^0_{k-2,k}) \\
&\quad +2(k-i)(J^0_{k-i,l}J^0_{l+i,k} - \delta_{i0}J^0_{k,k}) - 2(l+1+i)J^0_{k-1-i,l}J^0_{l+1+i,k} \\
&\quad +(k+2-i)(J^0_{k+2-i,l}J^0_{l+i,k} - \delta_{i0}J^0_{k+2,k}) - (l+3+i)J^0_{k-1-i,l}J^0_{l+3+i,k}\Big]\Bigg\} \\
&+\sum_{k=1}^{N}\Bigg\{\nu(N-k+1)(N-k+2)J^0_{k-2,k} - \Big[2k\nu(N-k) + 8\gamma k\Big]J^0_{k,k} \\
&\qquad -2(\alpha+2\beta)(N-k+1)J^0_{k-1,k} \\
&\qquad +\Big[\nu k(k+1) + 2(n-1)(k+2)\Big]J^0_{k+2,k}\Bigg\} \qquad (28) \\
&-2\sum_{k=1}^{N}J^0_{k+1,k}J^+_1 + 2(n+\nu+1)J^+_2 + 2(\alpha-2\beta)J^+_1 - 2\sum_{k=1}^{N}J^0_{k,k}J^+_2 - J^+_1J^+_1.
\end{aligned}
$$

Here we define $J^0_{0i} = J^-_i$ (i=1,2,...,N).

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Note added. After submission of this paper, an article[6] has been brought to the my attention. In that article, the complete classification of the quasi-exactly solvable models from the sl(M+1) generators are given. I am thankful to Professor Toshiaki Tanaka for informing his articles.

References

1. A. Turbiner, *Contemp.Math.* **Vol.160** 263-310 (1994).
2. A. Minzoni, M. Rosenbaum and A. Turbiner, *Mod.Phys.Lett.* **A11** 1977-1984 (1996).
3. L. Brink, A. Turbiner and N. Wyllard, *J.Math.Phys.* **39** 1285-1315 (1998).
4. Xinrui Hou and M. Shifman, *Int.J.Mod.Phys.* **A14** 2993-3004 (1999).
5. F. Finkel, D. Gómez-Ullate, A. González-Lopéz, M. A. Rodríguez and R. Zhdanov, *Comm.Math.Phys.* **221** 472-496 (2001).

6. Toshiaki Tanaka, *Annals Phys.* **309** 239-280 (2004).